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UNCERTAINTY IN A FISHERY MANAGEMENT GAME

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Uncertainty in a fishery management game *

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December, 2004

Abstract: In this paper we analyze the consequences of taking noise into account in a simple two-person fishery management game. Both a stochastic and deterministic formulation are considered. Compared to the noise-free model it is shown that the used stochastic framework has no implications for the equilibrium actions, whereas in the deterministic formulation as well the number of as the equilibrium actions themselves depend on the model parameters. The various equilibrium actions predicted using the deterministic framework seem to be quite plausible.

Keywords: linear quadratic differential games, feedback information structure, soft-constrained Nash equilibrium, infinite planning horizon.
Jel-codes: C61, C72, C73.

1 Introduction

Dynamic game theory brings together three features that are key to many situations in economy, ecology, and elsewhere: optimizing behavior, presence of multiple agents, and enduring consequences of decisions. For that reason this framework is often used to analyze various policy problems in these areas (see e.g. [6], [12] and [14]). Here we consider a fourth aspect, namely robustness with respect to variability in the environment. In the formulation of differential games, usually, a set of differential equations is specified including input functions that are controlled by the players, and players are assumed to optimize a criterion over time. The dynamic model is supposed to be an exact representation of the environment in which the players act; optimization takes place with no regard of possible deviations. It can safely be assumed, however, that agents in reality follow a different strategy. If an accurate model can be formed at all, it will in general be complicated and difficult to handle. Moreover it may be unwise to optimize on the basis of a too detailed model, in view of possible changes in dynamics that may take place in the course of time and that may be hard to predict. It makes more sense for agents to work on the basis of a relatively simple model and to look for strategies that are robust with respect to deviations between the model and reality. In an economic context, the importance of incorporating aversion to specification uncertainty has been stressed for instance by Hansen *et al.* [9].

In control theory, an extensive theory of robust design is already in place; see Başar [2] for a recent survey. We use this background to arrive at suitable ways of describing aversion to model risk in a dynamic game context. We assume linear dynamics and quadratic cost functions. These

*This paper has been used to prepare a subsection of the book [7].

assumptions are reasonable for situations of dynamic quasi-equilibrium, where no large excursions of the state vector are to be expected.

Following a pattern that has become standard in control theory two approaches will be considered. The first one is based on a stochastic approach. This approach assumes that the dynamics of the system are corrupted by a standard Wiener process (white-noise). Basic assumptions will be that the players have access to the current value of the state of the system and that the positive definite covariance matrix does not depend on the state of the system. Basically it turns out that under these assumptions the feedback Nash equilibria also constitute an equilibrium in such an uncertain environment. In the second approach, a malevolent disturbance input is introduced which is used in the modeling of aversion to specification uncertainty. That is, it is assumed that the dynamics of the system are corrupted by a deterministic noise component, and that each player has his own expectation about this noise. This is modeled by adapting for each player his cost function accordingly. The players cope with this uncertainty by considering a worst-case scenario. Consequently in this approach the equilibria of the game, in general, depend on the worst-case scenario expectations about the noise of the players.

The analysis is restricted to the infinite-planning horizon case. Furthermore only the feedback information structure is considered. For some results dealing with an open-loop information structure see e.g. [7, Chapter 7.4], [1], [13] and [11].

The outline of the paper is as follows. The next section formalizes the problem statement and summarizes the main theoretical results. In section three we analyze the various equilibrium outcomes for a simple fishery management problem. Section four concludes the paper with some general observations.

2 Preliminaries

In this paper we analyze a model that fits into the framework of a linear quadratic differential game. Below we will recall here the required theoretical results on these games for the two-player case. Proofs and the general N-player results can be found e.g. in [7, Chapters 8,9] or [3] and [4].

Throughout this paper we will use the shorthand notation

$$S_i := B_i R_{ii}^{-1} B_i^T \text{ and } S_{ij} := B_i R_{ii}^{-1} R_{ji} R_{ii}^{-1} B_i^T,\ i \neq j.$$

Furthermore we will assume everywhere that the players have full access to the current state of the system and that the set of control actions that are used by the players are the constant linear feedback strategies. That is,

$$u_i(t) = F_i x(t),\ \text{with } F_i \in \mathbb{R}^{m_i \times n},\ i = 1, 2,$$

and where (F_1, F_2) belongs to the set

$$\mathcal{F} := \{F = (F_1, F_2) \mid A + B_1 F_1 + B_2 F_2 \text{ is stable}\}.$$

The stabilization constraint is imposed to ensure the finiteness of the infinite-horizon cost integrals that we will consider. This assumption can also be justified from the supposition that one is studying a perturbed system which is temporarily out of equilibrium. In that case it is reasonable to expect that the state of the system remains close to the origin.

To make sure that our problem setting makes sense, we assume throughout this paper that the set $\mathcal{F}$ is non-empty. A necessary and sufficient condition for this to hold is that the matrix pair $(A, \begin{bmatrix} B_1, & B_2 \end{bmatrix})$ is stabilizable.

2.1 The noise-free case

Our benchmark noise-free case is the minimization of the performance criterion

$$J_i(x_0, u_1, u_2) = \lim_{T \to \infty} J_i(x_0, u_1, u_2, T) \tag{2.1.1}$$

for player i, $i = 1, 2$, where

$$J_i(x_0, u_1, u_2, T) = \int_0^T \left\{ x^T(t) Q_i x(t) + u_i^T(t) R_{ii} u_i(t) + u_j(t) R_{ij} u_j(t) \right\} dt, \; j \neq i, \tag{2.1.2}$$

subject to the dynamical system

$$\dot{x}(t) = Ax(t) + B_1 u_1(t) + B_2 u_2(t), \; x(0) = x_0. \tag{2.1.3}$$

Here Q_i and R_{ij}, $i, j = 1, 2$, are symmetric and R_{ii}, $i = 1, 2$, is positive definite. Notice that we do not make any definiteness assumptions w.r.t. Q_i.

The concept of a linear feedback Nash equilibrium on an infinite-planning horizon is then defined as follows.

Definition 2.1 $(F_1^*, F_2^*) \in \mathcal{F}$ is called a *stationary linear feedback Nash equilibrium* if the following inequalities hold:

$$J_1(x_0, F_1^*, F_2^*) \leq J_1(x_0, F_1, F_2^*) \text{ and } J_2(x_0, F_1^*, F_2^*) \leq J_2(x_0, F_1^*, F_2)$$

for each x_0 and for each state feedback matrix F_i, $i = 1, 2$ such that (F_1^*, F_2) and $(F_1, F_2^*) \in \mathcal{F}$. $\square$

Unless stated differently, the phrases "stationary" and "linear" in the above definition will be dropped here.

Next, consider the set of coupled algebraic Riccati equations

$$\begin{aligned} 0 &= -(A - S_2K_2)^T K_1 - K_1(A - S_2K_2) + K_1S_1K_1 - Q_1 - K_2S_{21}K_2, && (2.1.4) \\ 0 &= -(A - S_1K_1)^T K_2 - K_2(A - S_1K_1) + K_2S_2K_2 - Q_2 - K_1S_{12}K_1. && (2.1.5) \end{aligned}$$

Theorem 2.2 below states that feedback Nash equilibria are completely characterized by *stabilizing solutions* of (2.1.4,2.1.5). That is, by solutions (K_1, K_2) for which the closed-loop system matrix $A - S_1K_1 - S_2K_2$ is stable.

Theorem 2.2 *Let (K_1, K_2) be a stabilizing solution of (2.1.4,2.1.5) and define $F_i^* := -R_{ii}^{-1} B_i^T K_i$ for $i = 1, 2$. Then (F_1^*, F_2^*) is a feedback Nash equilibrium. Moreover, the cost incurred by player i by playing this equilibrium action is $x_0^T K_i x_0$, $i = 1, 2$.*
Conversely, if (F_1^*, F_2^*) is a feedback Nash equilibrium, there exists a stabilizing solution (K_1, K_2) of (2.1.4,2.1.5) such that $F_i^* = -R_{ii}^{-1} B_i^T K_i$. $\square$

2.2 Stochastic Approach

Next assume that the state of the system is generated by the linear noisy system,

$$\dot{x}(t) = Ax(t) + B_1u_1(t) + B_2u_2(t) + w(t). \tag{2.2.1}$$

The noise w is white, Gaussian, of zero mean and has covariance $V > 0$. The initial state at time $t = 0$, x_0, is a Gaussian random variable of mean m_0 and covariance P_0. This random variable is independent of w.

Before we introduce the considered performance criteria first notice that, at least when $Q > 0$, and one considers the with this game corresponding one-player stochastic regulator problem, the corresponding cost become unbounded if $T \to \infty$ (see e.g. Davis [5], pp.185). This is intuitively also clear, as the system is constantly perturbed by the noise w. For that reason we have to adapt the cost functionals for the players. Instead of minimizing the total cost, we will consider the minimization of the average cost per unit time:

$$L_i(V, u_1, \cdots, u_N) = \lim_{T\to\infty} \frac{1}{T}\mathrm{E}\{\int_0^T (x^TQ_ix + u_1^TR_{i1}u_1 + u_2^TR_{i2}u_2)dt\},\ i = 1, 2. \tag{2.2.2}$$

Given this context, we consider the next equilibrium concept.

Definition 2.3 $\hat{F} = (\hat{F}_1, \hat{F}_2) \in \mathcal{F}$ is called a *stochastic variance-independent feedback Nash equilibrium* if the following inequalities hold:

$$L_1(V, \hat{F}_1, \hat{F}_2) \le L_1(V, F_1, \hat{F}_2) \text{ and } L_2(V, \hat{F}_1, \hat{F}_2) \le L_2(V, \hat{F}_1, F_2)$$

for each $V \in \mathcal{V}$ and for each state feedback matrix F_i, $i = 1, 2$ such that $(\hat{F}_1, F_2)$ and $(F_1, \hat{F}_2) \in \mathcal{F}$. Here $\mathcal{V}$ is the set of all real positive semi-definite $n \times n$ matrices. □

Theorem 2.4 *Let (X_1, X_2) be a stabilizing solution of the algebraic Riccati equations (2.1.4,2.1.5) and define $\hat{F}_i := -R_{ii}^{-1}B_i^TX_i$ for $i = 1, 2$. Then $\hat{F} := (\hat{F}_1, \hat{F}_2)$ is a stochastic variance-independent feedback Nash equilibrium. Conversely, if $(\hat{F}_1, \hat{F}_2)$ is a stochastic variance-independent feedback Nash equilibrium, there exists a stabilizing solution (X_1, X_2) of (2.1.4,2.1.5) such that $\hat{F}_i = -R_{ii}^{-1}B_i^TX_i$. Moreover, $L_i(V, \hat{F}) = \mathrm{tr}(VX_i)$.* □

Theorem 2.4 shows that the linear feedback Nash equilibrium actions from Theorem 2.2 are also equilibrium actions for the stochastic game. Obviously, the corresponding cost differ. These cost are difficult to compare, since in the stochastic framework the average cost is used as performance criterion instead of the total cost.

2.3 Deterministic Approach

Our second approach to deal with uncertainty in the game assumes that the system is corrupted by some deterministic input. The considered dynamic model reads now as follows:

$$\dot{x}(t) = Ax(t) + B_1u_1(t) + B_2u_2(t) + Ew(t),\ x(0) = x_0. \tag{2.3.1}$$

Here $w \in L_2^q(0,\infty)$ is a q-dimensional disturbance vector affecting the system and E is a constant matrix.

The description of the players' objectives given above needs to be modified in order to express a desire for robustness. To that end, we modify the criterion (2.1.2) to

$$\bar{J}_i^{SC}(x_0, F_1, F_2) := \sup_{w \in L_2^q(0,\infty)} J_i(x_0, F_1, F_2, w) \tag{2.3.2}$$

where

$$J_i(x_0, F_1, F_2, w) := \int_0^\infty \{x^T(Q_i + F_1^T R_{i1} F_1 + F_2^T R_{i2} F_2)x - w^T V_i w\} dt. \tag{2.3.3}$$

The weighting matrix V_i is symmetric and positive definite for both $i = 1, 2$. Because it occurs with a minus sign in (2.3.3), this matrix constrains the disturbance vector w in an indirect way so that it can be used to describe the aversion to model risk of player i. Specifically, if the quantity $w^T V_i w$ is large for a vector $w \in \mathbb{R}^q$, this means that player i does not expect large deviations of the nominal dynamics in the direction of Ew. Furthermore, the larger he chooses V_i, the closer the worst case signal he can be confronted with in this model will approach the zero input signal (that is: $w(.) = 0$). In line with the nomenclature used in control theory literature we will call this the "soft-constrained" formulation.

The equilibrium concept that will be used in this deterministic setting is based on the adjusted cost functions (2.3.2). A formal definition is given below.

Definition 2.5 $\overline{F} = (\overline{F}_1, \overline{F}_2) \in \mathcal{F}$ is called a *soft-constrained Nash equilibrium* if the following inequalities hold:

$$\overline{J}_1(x_0, \overline{F}_1, \overline{F}_2) \leq \overline{J}_1(x_0, F_1, \overline{F}_2) \text{ and } \overline{J}_2(x_0, \overline{F}_1, \overline{F}_2) \leq \overline{J}_2(x_0, \overline{F}_1, F_2)$$

for each x_0 and for each state feedback matrix $F_i,\ i = 1, 2$ such that $(\overline{F}_1, F_2)$ and $(F_1, \overline{F}_2) \in \mathcal{F}$. $\square$

Using the shorthand notation

$$M_i := E V_i^{-1} E^T,$$

we have the next result.

Theorem 2.6 *Consider the differential game defined by (2.3.1–2.3.2). Assume there exist real symmetric $n \times n$ matrices $X_i,\ i = 1, 2$, and real symmetric $n \times n$ matrices $Y_i,\ i = 1, 2$, such that*

$$-(A - S_2X_2)^T X_1 - X_1(A - S_2X_2) + X_1S_1X_1 - Q_1 - X_2S_{21}X_2 - X_1M_1X_1 = 0, \tag{2.3.4}$$

$$-(A - S_1X_1)^T X_2 - X_2(A - S_1X_1) + X_2S_2X_2 - Q_2 - X_1S_{12}X_1 - X_2M_2X_2 = 0, \tag{2.3.5}$$

$$A - S_1X_1 - S_2X_2 + M_1X_1 \text{ and } A - S_1X_1 - S_2X_2 + M_2X_2 \text{ are stable,} \tag{2.3.6}$$

$$A - S_1X_1 - S_2X_2 \text{ is stable} \tag{2.3.7}$$

$$-(A - S_2X_2)^T Y_1 - Y_1(A - S_2X_2) + Y_1S_1Y_1 - Q_1 - X_2S_{21}X_2 \leq 0, \tag{2.3.8}$$

$$-(A - S_1X_1)^T Y_2 - Y_2(A - S_1X_1) + Y_2S_2Y_2 - Q_2 - X_1S_{12}X_1 \leq 0. \tag{2.3.9}$$

Define $\overline{F} = (\overline{F}_1, \overline{F}_2)$ *by*

$$\overline{F}_i := -R_{ii}^{-1} B_i^T X_i, \ i = 1, 2.$$

Then $\overline{F} \in \mathcal{F}$*, and* $\overline{F}$ *is a soft-constrained Nash equilibrium. Furthermore, the worst-case signal* $\bar{w}_i$ *from player* i'*s perspective is*

$$\bar{w}(t) = V_i^{-1} E^T X_i e^{(A - S_1 X_1 - S_2 X_2 + M_i X_i)t} x_0.$$

Moreover the cost for player i *under the realization of his worst-case expectations are*

$$\overline{J}_i^{SC}(\overline{F}_1, \overline{F}_2, x_0) = x_0^T X_i x_0, \ i = 1, 2.$$

Conversely, if $(\bar{F}_1, \bar{F}_2)$ *is a soft-constrained Nash equilibrium, the equations (2.3.4–2.3.7) have a set of real symmetric solutions* (X_1, X_2)*.*

Corollary 2.7 *If* $Q_i \geq 0, \ i = 1, 2$ *and* $S_{ij} \geq 0, \ i, j = 1, 2$*, the matrix inequalities (2.3.8–2.3.9) are trivially satisfied with* $Y_i = 0, \ i = 1, 2$*. So, under these conditions the differential game defined by (2.3.1–2.3.2) has a soft-constrained Nash equilibrium if and only if the equations (2.3.4–2.3.7) have a set of real symmetric* $n \times n$ *matrices* $X_i, \ i = 1, 2$*.* □

Remark 2.8 This deterministic formulation allows also for a stochastic interpretation. This interpretation can be given based on the well known connection between the H_∞ control problem and the risk sensitive Linear Exponential Quadratic Gaussian (LEQG) control problem (see e.g. [2], [1, Section 4.7], [8], [10], [15] and [17]). Details on this can be found e.g. in [7].

3 A fishery management game

This section illustrates some consequences of taking deterministic noise into account by means of a simple fishery management problem.

Consider two fishermen who fish a lake. Let s be the number of fish in the lake. Assume that the price $p(t)$ the fishermen get for their fish is fixed, i.e.,

$$p(t) = p.$$

The growth of the fish stock in the lake is described by

$$\dot{s}(t) = \beta s(t) - u_1(t) - u_2(t) - w(t), \ s(0) = s_0 > 0$$

where w is a factor which has a negative impact on the growth of the fish stock (e.g. water pollution, weather, birds, local fishermen etc.). Both fishermen have their own expectations about the consequences of these negative influences on the fish growth and cope with this by considering the next optimization problem

$$J_i := \min_{u_i \in \mathcal{F}^{aff}} \sup_{w \in L_2} \int_0^\infty e^{-rt} \{-p u_i(t) + \gamma_i u_i^2(t) - v_i w^2(t)\} dt, \ i = 1, 2,$$

where

$$\mathcal{F}^{aff} := \{(u_1, u_2) \mid u_i(t) = f_{ii}s(t) + g_i, \text{ with } \beta - f_{11} - f_{22} < \frac{1}{2}r\}.$$

In this formulation all constants, $r, \alpha_i, \beta, \gamma_i$ and v_i, are positive. The term $\gamma_i u_i^2$ models the cost involved for fisherman i in catching an amount u_i of fish. We will assume that $v_i > \gamma_i,\ i = 1, 2$. That is, each fisherman does not expect that a situation will occur where the deterministic cost will be larger than his normal cost of operation. Notice that, since in this formulation the involved cost for the fishermen depends quadratically on the amount of fish they catch, catching large amounts of fish is not profitable for them. This observation might model the fact that catching a large amount of fish is, from a practical point of view, impossible for them. This might be due to either technical restrictions and/or the fact that there is not an abundant amount of fish in the lake. That is, catching much more fish requires much more advanced technology which costs rise quadratically.

Introducing $x^T(t) := [s(t)\ 1]$, the optimization problem can be rewritten as

$$\min_{u_i \in \mathcal{F}^{aff}} \sup_{w \in L_2(0,\infty)} \int_0^\infty e^{-rt}\{[x^T(t)\ u_i^T(t)] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}p \\ 0 & -\frac{1}{2}p & \gamma_i \end{bmatrix} \begin{bmatrix} x(t) \\ u_i(t) \end{bmatrix} - v_i w^2(t)\}dt,\ i = 1, 2,$$

subject to the dynamics

$$\dot{x}(t) = \begin{bmatrix} \beta & 0 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} u_1(t) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} u_2(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(t),\ x(0) = \begin{bmatrix} s_0 \\ 1 \end{bmatrix}.$$

Using the transformation

$$\tilde{u}_i := u_i - \frac{p}{2\gamma_i},\ i = 1, 2,$$

the optimization problem can be rewritten as

$$\min_{\tilde{u}_i \in \mathcal{F}^{aff}} \sup_{w \in L_2(0,\infty)} \int_0^\infty e^{-rt}\{\tilde{x}^T(t) \begin{bmatrix} 0 & 0 \\ 0 & \frac{-p^2}{4\gamma_i} \end{bmatrix} \tilde{x}(t) + \gamma_i \tilde{u}_i^2(t) - v_i w^2(t)\}dt,\ i = 1, 2,$$

subject to the dynamics

$$\dot{\tilde{x}}(t) = \begin{bmatrix} \beta & \frac{-p}{2\gamma_1} + \frac{-p}{2\gamma_2} \\ 0 & 0 \end{bmatrix} \tilde{x}(t) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \tilde{u}_1(t) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \tilde{u}_2(t) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} w(t),\ \tilde{x}(0) = \begin{bmatrix} s_0 \\ 1 \end{bmatrix}.$$

With, $\hat{x}(t) := e^{-\frac{1}{2}rt}\tilde{x}(t)$, $\hat{u}_i(t) := e^{-\frac{1}{2}rt}\tilde{u}_i(t),\ i = 1, 2$, and $\hat{w}(t) := e^{-\frac{1}{2}rt}w(t)$, we obtain the standard formulation.

$$\min_{\hat{u}_i \in \mathcal{F}} \sup_{\hat{w} \in L_2(0,\infty)} \int_0^\infty \{\hat{x}^T(t) \begin{bmatrix} 0 & 0 \\ 0 & \frac{-p^2}{4\gamma_i} \end{bmatrix} \hat{x}(t) + \gamma_i \hat{u}_i^2(t) - v_i \hat{w}^2(t)\}dt,\ i = 1, 2,$$

subject to the dynamics

$$\dot{\hat{x}}(t) = \begin{bmatrix} \beta - \frac{1}{2}r & \frac{-p}{2\gamma_1} + \frac{-p}{2\gamma_2} \\ 0 & -\frac{1}{2}r \end{bmatrix} \hat{x}(t) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \hat{u}_1(t) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \hat{u}_2(t) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \hat{w}(t),\ \hat{x}(0) = \begin{bmatrix} s_0 \\ 1 \end{bmatrix}.$$

According Theorem 2.6 the soft-constrained Nash equilibria for this game are obtained as

$$\hat{u}_i(t) = [\frac{1}{\gamma_i}\ 0]X_i\hat{x}(t),$$

where with

$$A := \begin{bmatrix} \beta - \frac{1}{2}r & \frac{-p}{2\gamma_1} + \frac{-p}{2\gamma_2} \\ 0 & -\frac{1}{2}r \end{bmatrix};\ S_i := \begin{bmatrix} \frac{1}{\gamma_i} & 0 \\ 0 & 0 \end{bmatrix};\ S_{ij} = 0;\ M_i := \begin{bmatrix} \frac{1}{v_i} & 0 \\ 0 & 0 \end{bmatrix};\ Q_i := \begin{bmatrix} 0 & 0 \\ 0 & \frac{-p^2}{4\gamma_i} \end{bmatrix};$$

(X_1, X_2) solve (2.3.4,2.3.5) and satisfy the conditions (2.3.6–2.3.7). Notice that with

$$Y_i := \begin{bmatrix} 0 & 0 \\ 0 & \frac{-p^2}{4r\gamma_i} \end{bmatrix},\ i = 1, 2,$$

the inequalities (2.3.8–2.3.9) are satisfied.

In case the discount factor, r, is more than two times larger than the exogenous growth rate, β, of the fish population we see by straightforward substitution that

$$X_i := \begin{bmatrix} 0 & 0 \\ 0 & \frac{-p^2}{4r\gamma_i} \end{bmatrix},\ i = 1, 2, \tag{3.0.10}$$

satisfy the equations (2.3.4–2.3.7). So one soft-constrained Nash equilibrium, in that case, is provided by

$$u_i^*(t) = \frac{p}{2\gamma_i}. \tag{3.0.11}$$

That is, irrespective of the growth of the fish population, the fishermen catch a constant amount of fish each time. This amount is completely determined by their cost function and the price of the fish.

To see whether there exist still different equilibria, we elaborate the equations (2.3.4–2.3.5). To that end introduce

$$X_1 =: \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} \text{ and } X_2 =: \begin{bmatrix} z_1 & z_2 \\ z_2 & z_3 \end{bmatrix}.$$

Then, the equations (2.3.4–2.3.5) can be rewritten as

$$(-2\beta + r + \frac{2}{\gamma_2}z_1)x_1 + (\frac{1}{\gamma_1} - \frac{1}{v_1})x_1^2 = 0 \tag{3.0.12}$$

$$(\frac{p}{2\gamma_1} + \frac{p}{2\gamma_2} + \frac{1}{\gamma_2}z_2)x_1 + (r - \beta + \frac{1}{\gamma_2}z_1)x_2 + (\frac{1}{\gamma_1} - \frac{1}{v_1})x_1x_2 = 0 \tag{3.0.13}$$

$$\frac{p^2}{4\gamma_1} + (\frac{p}{\gamma_1} + \frac{p}{\gamma_2} + \frac{2}{\gamma_2}z_2)x_2 + (\frac{1}{\gamma_1} - \frac{1}{v_1})x_2^2 + rx_3 = 0 \tag{3.0.14}$$

$$(-2\beta + r + \frac{2}{\gamma_1}x_1)z_1 + (\frac{1}{\gamma_2} - \frac{1}{v_2})z_1^2 = 0 \tag{3.0.15}$$

$$(\frac{p}{2\gamma_1} + \frac{p}{2\gamma_2} + \frac{1}{\gamma_1}x_2)z_1 + (r - \beta + \frac{1}{\gamma_1}x_1)z_2 + (\frac{1}{\gamma_2} - \frac{1}{v_2})z_1z_2 = 0 \tag{3.0.16}$$

$$\frac{p^2}{4\gamma_2} + (\frac{p}{\gamma_1} + \frac{p}{\gamma_2} + \frac{2}{\gamma_1}x_2)z_2 + (\frac{1}{\gamma_2} - \frac{1}{v_2})z_2^2 + rz_3 = 0. \tag{3.0.17}$$

From the first equation (3.0.12) it follows that either

$$(i)\ x_1 = 0 \text{ or } (ii)\ (\frac{1}{\gamma_1} - \frac{1}{v_1})x_1 + \frac{2}{\gamma_2}z_1 = 2\beta - r.$$

In case (i), $x_1 = 0$, (3.0.13,3.0.15) yields that $x_2 = 0$ (under the assumptions that $\beta \neq r$ and $\beta + \frac{\gamma_2}{v_2}(\beta - r) \neq 0$). Equation (3.0.14) shows then that necessarily $x_3 = -\frac{p^2}{4r\gamma_1}$. Equations (3.0.15–3.0.17) give then that either X_2 is as reported in (3.0.10) or

$$X_2 = \begin{bmatrix} \frac{\gamma_2 v_2(2\beta-r)}{v_2-\gamma_2} & \bar{x}_2 \\ \bar{x}_2 & -\frac{p^2}{4r\gamma_2} - \frac{p}{2\beta}(\frac{1}{\gamma_1} + \frac{1}{\gamma_2})\bar{x}_2 \end{bmatrix}. \tag{3.0.18}$$

Here $\bar{x}_2 = \frac{v_2\gamma_2}{v_2-\gamma_2}c$, with

$$c := -\frac{p(\gamma_1 + \gamma_2)(2\beta - r)}{2\beta\gamma_1\gamma_2}.$$

Similarly, a lengthy analysis of case (ii) shows that besides the solutions (3.0.10,3.0.18) this set of equations has (given our parametric assumptions) still two other solutions. Introducing, for notational convenience,

$$d := (v_1 - \gamma_1)(v_2 + \gamma_2) + (v_1 + \gamma_1)(v_2 - \gamma_2) + (v_1 + \gamma_1)(v_2 + \gamma_2),$$

these solutions are

$$X_1 := \begin{bmatrix} \frac{\gamma_1 v_1(2\beta-r)}{v_1-\gamma_1} & \bar{y}_2 \\ \bar{y}_2 & -\frac{p^2}{4r\gamma_1} - \frac{p}{2\beta}(\frac{1}{\gamma_1} + \frac{1}{\gamma_2})\bar{y}_2 \end{bmatrix},\ X_2 = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{p^2}{4r\gamma_2} \end{bmatrix}, \tag{3.0.19}$$

where $\bar{y}_2 = \frac{v_1\gamma_1}{(v_1-\gamma_1)}c$; and

$$X_1 := \begin{bmatrix} \frac{\gamma_1 v_1(\gamma_2+v_2)(2\beta-r)}{d} & \bar{z}_2 \\ \bar{z}_2 & -\frac{p^2}{4r\gamma_1} - \frac{p}{2\beta}(\frac{1}{\gamma_1} + \frac{1}{\gamma_2})\bar{z}_2 \end{bmatrix}, X_2 := \begin{bmatrix} \frac{\gamma_2 v_2(\gamma_1+v_1)(2\beta-r)}{d} & \tilde{z}_2 \\ \tilde{z}_2 & -\frac{p^2}{4r\gamma_2} - \frac{p}{2\beta}(\frac{1}{\gamma_1} + \frac{1}{\gamma_2})\tilde{z}_2 \end{bmatrix}, \tag{3.0.20}$$

where $\bar{z}_2 = \frac{v_1\gamma_1(v_2+\gamma_2)}{d}c$ and $\tilde{z}_2 = \frac{v_2\gamma_2(v_1+\gamma_1)}{d}c$.
From (3.0.18–3.0.20) the next potential equilibrium actions result

$$(u_1^*(t), u_2^*(t)) = (\frac{p}{2\gamma_1}, \frac{v_2}{v_2 - \gamma_2}((2\beta - r)s(t) + c) + \frac{p}{2\gamma_2}); \tag{3.0.21}$$

$$(u_1^*(t), u_2^*(t)) = (\frac{v_1}{v_1 - \gamma_1}((2\beta - r)s(t) + c) + \frac{p}{2\gamma_1}, \frac{p}{2\gamma_2}); \tag{3.0.22}$$

$$(u_1^*(t), u_2^*(t)) = (\frac{v_1(\gamma_2 + v_2)}{d}((2\beta - r)s(t) + c) + \frac{p}{2\gamma_1}, \frac{v_2(\gamma_1 + v_1)}{d}((2\beta - r)s(t) + c) + \frac{p}{2\gamma_2}), \tag{3.0.23}$$

respectively.
To see whether they actually can arise as equilibria we have to verify whether there are parametric conditions such that the stability constraints (2.3.6) and (2.3.7) are met. Straightforward calculations

show that for each of these three equilibria (3.0.21–3.0.23) these stability conditions are satisfied if and only if $2\beta - r > 0$. As an example consider the equilibrium strategy (3.0.23). Using this strategy,

$$A - S_1X_1 - S_2X_2 = \begin{bmatrix} -\frac{(2\beta-r)(v_1+\gamma_1)(v_2+\gamma_2)}{2d} & h_1 \\ 0 & -\frac{1}{2}r \end{bmatrix},$$

whereas

$$A - S_1X_1 - S_2X_2 + M_iX_i = \begin{bmatrix} -\frac{(2\beta-r)(v_i-\gamma_i)(v_j+\gamma_j)}{2d} & h_2 \\ 0 & -\frac{1}{2}r \end{bmatrix},\ i,j = 1,2,\ j \neq i.$$

Here, $h_i,\ i = 1,2,3$, are some parameters which are not important for the stability analysis. The claim follows then directly by considering the first entry of all these matrices. All these entries are negative if and only if $2\beta - r > 0$.

From this analysis it follows that if $r > 2\beta$ the game has a unique equilibrium (3.0.11) which is characterized by fishing a fixed amount of fish by both fishermen. Due to the large discounting rate the players seem to be indifferent to their noise expectations. This, since the fixed amount of fish they catch is independent of these expectations. This equilibrium results in a situation where, under the assumptions that the initial fish stock s_0 is larger than $\frac{p}{2\beta}(\frac{1}{\gamma_1} + \frac{1}{\gamma_2})$ and the deterministic negative impact factor w is not too large, the amount of fish will grow steadily with a factor β. Notice that the expected worst-case revenues (i.e. $-J_i^*$) of fisherman i are $-\hat{x}^T(0)X_i\hat{x}(0) = \frac{p^2}{4r\gamma_i},\ i = 1,2$. This coincides with the actual revenues obtained using these actions, as measured by

$$-\int_0^\infty e^{-rt}\{-pu_i(t) + \gamma_i u_i^2(t)\}dt,\ i = 1,2.$$

In case $r < 2\beta$ a different situation occurs. Then, three different equilibria occur. Two equilibria correspond with a situation where one fisherman fishes a fixed amount of fish, whereas the amount of fish the other fisherman catches consists of a fixed amount (that might be negative, which can be interpreted as that the fisherman plants some fish), and an additional amount that depends on both the fishstock and his expectations about the deterministic disturbance. In the third equilibrium both fishermen catch an amount of fish that depends on the fishstock additional to some fixed (possibly negative) amount.

In case $g_i := v_i(r - \beta) - \beta\gamma_i < 0$ and the external factors w are modest, the fish stock will converge to some fixed amount in the first two equilibria (3.0.21) and (3.0.22), respectively. This amount depends on the actual realization of the external factor w. In case $g_i > 0$, on the other hand, the fish stock will grow steadily with a growth factor g provided $s_0 > \frac{p(\gamma_1+\gamma_2)}{2\beta\gamma_1\gamma_2(v_i-\gamma_i)},\ i = 1,2$ (assuming again that the external factors are not too unwildy). The expected worst-case revenues of one of the fishermen, i, in these two equilibria are $\frac{p^2}{4r\gamma_i}$ (which coincide again with his actual obtained revenues), whereas the revenues under the worst-case realization of the noise for fisherman j are

$$-J_j^* = -\hat{x}_0^T\left(\frac{\gamma_j}{v_j - \gamma_j}H_{1,j} + H_{2,j}\right)\hat{x}_0, \tag{3.0.24}$$

where

$$H_{1,j} = \begin{bmatrix} \gamma_j(2\beta - r) & c\gamma_j \\ c\gamma_j & -\frac{pc\gamma_j}{2\beta}(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}) \end{bmatrix} \text{ and } H_{2,j} = \begin{bmatrix} \gamma_j(2\beta - r) & c\gamma_j \\ c\gamma_j & -\frac{p^2}{4r\gamma_j} - \frac{pc\gamma_j}{2\beta}(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}) \end{bmatrix}$$

$i,j=1,2,\ j\neq i$. Since the amount of fish catched by this fisherman now depends on the fish stock, and thus in particular on the realization of the disturbance factor w, in general these worst-case cost will differ from the actual revenues for him.
Notice that both $H_{1,j}$ and $H_{2,j}$ do not depend on the noise parameters and that $H_{1,j}$ is positive semi-definite. Consequently, $\frac{\gamma_j}{v_j-\gamma_j}H_{1,j}$ in (3.0.24) reflects the for the fisherman involved cost due to his (worst-case) expectations concerning the external factor w. This effect is almost negligible if the fisherman expects a modest influence of w on the fish growth (i.e. if v_i is at least a number of times larger than γ_i). In that case the cost J_i^* are close to the cost the fisherman has in the undisturbed case ($v_i=\infty$). In case the fisherman's worst-case expectations about w are large (i.e. v_i close to γ_i) these worst-case expected cost are completely dominated by $\frac{\gamma_j}{v_j-\gamma_j}H_{1,j}$. It seems reasonable to take the view that a fisherman will only go fishing (assuming that he wants to have a profit even under his worst-case expectations about w) if $-J_i^*$ is positive. This gives additional conditions on the parameters that have to be satisfied to consider this equilibrium outcome as a realistic one. Particularly when there is a very large initial fish stock, these conditions will usually not be satisfied. But given our model assumptions this is a situation which we can rule out. Also, in case the expected external factors w become dominating generically the revenues $-J_i^*$ become negative. This, since $H_{1,j}$ in (3.0.24) is positive semi-definite. So, also in that case the equilibrium ceases to exist. Again this case is intuitively clear. If there will be a large impact of "external fishing", almost no fish is left in the lake. So the fisherman is confronted with exceptional cost to catch the remainder of the fish. Since he gets a fixed price on the market for his fish he will quit fishing.
Finally, notice that both $\frac{\partial u_j^*}{\partial v_j}<0$ and $\frac{\partial J_j^*}{\partial v_j}<0$. From this, one conclusion is that the more fisherman j expects that the fish stock will be disturbed by external factors, the more fish he will catch himself. Another conclusion is that the expected returns under the worst-case scenario decrease for a fisherman if he expects more negative external impacts.

In the third equilibrium (3.0.23) the cost for fisherman j are:

$$J_j^*=\hat{x}^T(0)\left(h_jH_{1,j}+\begin{bmatrix}\frac{\gamma_j(2\beta-r)}{3} & \frac{c\gamma_j}{3}\\ \frac{c\gamma_j}{3} & -\frac{p^2}{4r\gamma_j}-\frac{pc\gamma_j}{6\beta}(\frac{1}{\gamma_1}+\frac{1}{\gamma_2})\end{bmatrix}\right)\hat{x}(0), \tag{3.0.25}$$

$i,j=1,2,\ j\neq i$, where $h_j=\frac{2\gamma_i v_j-\gamma_j v_i+\gamma_1\gamma_2}{3d}$. For this equilibrium a similar analysis can be performed as above, yielding similar conclusions. We will not elaborate those points here.
One point in which this equilibrium differs from the previous one is that the equilibrium action and expected worst-case revenues depend on the opponent fisherman's noise expectations. From (3.0.23) and (3.0.25) it follows that both $\frac{\partial u_j^*}{\partial v_i}$ and $\frac{\partial J_j^*}{\partial v_i}$ are negative, whereas both $\frac{\partial u_j^*}{\partial v_j}$ and $\frac{\partial J_j^*}{\partial v_j}$ are positive. This implies that each fisherman responds to an increase in worst-case expectations about the external factors of the other fisherman by catching more fish. Furthermore, an increase of the worst-case expectations of his colleague has a negative impact on his worst-case expected revenues. This, contrary to his own reaction to an increase in worst-case expectations w.r.t. external factors. If the fisherman expects himself more "disturbances" he will react by catching less fish which has a positive impact on his worst-case expected revenues.

Finally, straightforward calculations show that no one of the three equilibria Pareto dominates another equilibrium. That is, comparing the worst-case revenues of any two of the above three equilibria always one fisherman is better of in one equilibrium whereas the other is better of in the other equilibrium.

For completeness we summarized in Tables 1 and 2 the various equilibria for the noise-free (c.q.

Case $\mathbf{r} > \mathbf{2}\beta$	Revenues	u_i^*
F-man 1	d_1	$\bar{u}_1$
F-man 2	d_2	$\bar{u}_2$

Table 1: Noise-free Equilibrium revenues and actions if $\mathbf{r} > \mathbf{2}\beta$

Case $\mathbf{r} < \mathbf{2}\beta$	I. Revenues	u_i^*	II. Revenues	u_i^*	III. Revenues	u_i^*
F-man 1	$\frac{-k}{3}\gamma_1 + d_1$	$\frac{1}{3}u_f(t) + \bar{u}_1$	d_1	$\bar{u}_1$	$-k\gamma_1 + d_1$	$u_f(t) + \bar{u}_1$
F-man 2	$\frac{-k}{3}\gamma_2 + d_2$	$\frac{1}{3}u_f(t) + \bar{u}_2$	$-k\gamma_2 + d_2$	$u_f(t) + \bar{u}_2$	d_2	$\bar{u}_2$

Table 2: Noise-free Equilibrium revenues and actions if $\mathbf{r} < \mathbf{2}\beta$

stochastic) case. These tables are obtained by considering $v_i \to \infty$ in the above analysis. In these tables we used the notation $d_i := \frac{p^2}{4r\gamma_i}$, $\bar{u}_i := \frac{p}{2\gamma_i}$, $k := (2\beta - r)(s_0 + \frac{c}{2\beta - r})^2$, and $u_f(t) := (2\beta - r)s(t) + c$. Notice again that if the parameters are such that the revenues become negative, one may expect that the corresponding equilibrium will cease to exist. Taking this into account, we see from Table 1, that if $r > 2\beta$ there will always be a unique equilibrium. If, on the other hand $r < 2\beta$, we observe from Table 2 that the equilibrium combinations (I,II,III), (I,II), (I,III), (I), (II), (III) or none may occur.

4 Concluding Remarks

In conclusion we observe that taking noise into account has a number of consequences. First, in case $r > 2\beta$, noise does not affect the outcome of the game. The fishermen keep fishing a fixed amount over time.

If $r < 2\beta$ the noise expectations do play a role. Three different equilibria may occur. Two equilibria in which either one of the fishermen sticks to the noise-free optimal action and the other takes into account the current fish stock and his worst-case expectations about the external factors in the amount of fish he catches. At the other equilibrium both fishermen take simultaneously each other's noise expectations into account.

In all these equilibria we frequently observe a tragedy of common's effect. That is: the reaction by the fishermen to an expected more disturbed fish stock growth by either himself or his opponent is to increase the number of fish they catch. Only in the last-mentioned equilibrium a reverse reaction occurs: if a fisherman anticipates here a more disturbed environment he will catch less himself. This effect is however crossed if he also observes an increased sensitivity of his colleague w.r.t. the external factors.

Finally we observe that, assuming that the noise-free case has three equilibria, increasing the noise level has as a consequence that the number of equilibria gradually declines from three to none.

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